Functional approach for quantum systems with continuous spectrum

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Considering quantum states as functionals acting on observables to give their mean values, it is possible to deal with quantum systems with continuous spectrum, generalizing the concept of trace. Generalized observables and states are defined for a quantum oscillator linearly coupled to a scalar field, and the analytic expression for time evolution is obtained. The ''final'' state ($t\to\infty$) is presented as a weak limit. Finite and infinite numbers of excited modes of the field are considered. [S1063-651X(98)09004-7]

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I. INTRODUCTION

There is a great deal of interest in the search for a physical explanation for the evolution towards equilibrium of quantum systems in quantum statistical mechanics. For many years a great number of papers were devoted to this problem. As it is almost impossible to quote them all, let us only give some examples.

Standard results on dynamical systems show the impossibility that a system of linearly coupled oscillators will reach statistical equilibrium. This is due to the fact that the problem can be reduced to a set of noninteracting oscillators (the normal modes). As there is no mechanism for transferring energy between normal modes, the initial distribution of normal modes is not modified by time evolution.

In his pioneering work, Fermi *et al.* [1] showed that nonlinear couplings in general do not determine the approach to equilibrium. Prigogine [2] showed that for a very big set of coupled oscillators, a weak nonlinear part of the interaction produces the approach to equilibrium.

The problem of a chain of independent quantum oscillators with a linear coupling between first neighbors was analyzed by Blaise *et al.* [3]. Numerical experiments show that for the initial condition of a single excited oscillator the system evolves first towards equilibrium (equipartition of energy), and later a recurrence time appears for which the energy returns to the oscillator of the initial condition. The recurrence time grows proportionally to the number of oscillators.

A line of research [4–6] studied the quantum Brownian motion with path integrals, i.e., the motion of a quantum oscillator linearly coupled to a thermal bath.

This problem was also analyzed by Gruver *et al.* [7] by numerical methods, showing that a quantum ''big'' oscillator linearly coupled with a great number of identical ''small'' oscillators with a thermal distribution, evolving in such a way that it is possible to adjust its time evolution with an exponential that leads the ''big'' oscillator subsystem to the thermal equilibrium with the bath.

The works of Refs. [3], [6], and [7] deal with linear couplings for which the total Hamiltonian can be reduced to noninteracting normal modes.

The microscopic explanation of the approach to equilibrium was related also to the so-called ''intrinsic irreversibility'' of quantum systems. Misra *et al.* [8,9] pointed out the existence of a time operator for the statistical description of classical and quantum systems. The mean value of this operator is the ''age'' of the system, which is a growing function of time. Bohm *et al.* [10,11] related the intrinsic irreversibility to the existence of generalized eigenvectors of the Hamiltonian with complex eigenvalues, corresponding to poles of the analytic extension of the scattering matrix. Complex eigenvalues have been obtained by Sudarshan [12] by analytic continuation in a generalized quantum mechanics. Intrinsic irreversibility appears also through subdynamic theory [13,14].

When it is necessary to deal with systems with a huge number of particles, the standard procedure is to start with N particles in a box of volume V, making the limit $N\to\infty$, $V\to\infty$ with $N/V=c<\infty$ in the last step of the calculations. Even for a finite number of particles, the limit $V\to\infty$ produces a continuum spectrum in the unperturbed Hamiltonian. The usual formalism of quantum mechanics cannot be used in this case, due to the appearance of diagonal singularities in states and observables [15–17].

The Friedrichs model, describing the interaction between a quantum oscillator and a scalar field in the one excited mode sector, was extensively analyzed in the literature for the one excited mode sector. It is an exactly solvable model, in which the quantum oscillator decays to the ground state for all initial conditions. Sudarshan *et al.* [12] computed the complex spectral decomposition in the framework of a generalized quantum mechanics. The spectral decomposition was also obtained by Petrosky *et al.* [18] using subdynamic theory. The spectral decomposition with complex eigenvalues was interpreted in terms of Rigged-Hilbert spaces in Refs. [19] and [13].

Based in the pioneering work of Segal [20], Antoniou

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et al. [15,16] developed a formalism for quantum systems with diagonal singularity. The quantum states of this theory are *functionals* over certain space of observables $\mathcal{O}$. Mathematically this means that the space $\mathcal{S}$ of states is contained in $\mathcal{O}^{\times}$. Physically it means that the only thing we can really observe and measure are the mean values of the observables $O\in\mathcal{O}$ in states $\rho\in\mathcal{S}\subset O^{\times}$: namely, $\langle O\rangle_{\rho}=\rho[O]\equiv(\rho|O)$. This is the natural generalization of the usual trace $\mathrm{Tr}(\hat{\rho}\hat{O})$, which is ill defined in systems with continuous spectrum.

In this work we apply this formalism to a quantum oscillator linearly coupled to a scalar field, for a finite number of excited modes (decay process) and for the thermodynamic limit (infinite number of excited modes of the field). We will be able to obtain the time evolution of a finite number of excited modes, and to give an analytic expression for the "final" state ($t\to\infty$) as a weak limit on the "test observables" $\mathcal{O}$. As we shall see, the formalism can also be adapted to obtain exact expressions for the time evolution in the thermodynamic limit, where an infinite number of excited modes of the field drives the quantum oscillator to a final excited state. These results will be obtained without using coarse graining, complex generalized eigenvalues, or box normalization.

The model is presented in Sec. II. In Sec. III we discuss the characterization of states and observables with diagonal singularity, both for a finite and infinite number of excited modes. In Sec. IV we obtain the exact solution of the problem using the diagonalized form of the Hamiltonian. The master equation is solved in Sec. V, and their solutions are compared with the exact expressions of Sec. IV.

II. THE MODEL

We consider a quantum oscillator with the Hamiltonian

$$H_S=\Omega b^{\dagger}b,\quad [b,b^{\dagger}]=1,\quad (\hbar=1), \tag{1}$$

and a quantum field with the Hamiltonian

$$H_F=\int d\mathbf{k}\,\omega_k a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}},\quad [a_{\mathbf{k}},a^{\dagger}_{\mathbf{k}'}]=\delta^3(\mathbf{k}-\mathbf{k}'),$$

$$\omega_k=k\equiv|\mathbf{k}|,\quad (c=1). \tag{2}$$

The interaction is given by

$$H_{\rm int}=\int d\mathbf{k}\;V_k[a^{\dagger}_{\mathbf{k}}b+b^{\dagger}a_{\mathbf{k}}],\quad V_k^{*}=V_k,$$

$$[a^{\dagger}_{\mathbf{k}},b^{\dagger}]=[a^{\dagger}_{\mathbf{k}},b]=0. \tag{3}$$

The function V_k is chosen in such a way that

$$\eta_{\pm}(\omega_k)\equiv\omega_k-\Omega-\int\frac{d\mathbf{k}'V^2_{k'}}{\omega_k-\omega_{k'}\pm i0} \tag{4}$$

does not vanish for any $k\in\mathbb{R}^{+}$, and the analytic extension $\eta_{+}(z)$ from the upper to the lower complex half plane of $\eta_{+}(k)$ has a simple zero at $z=z_0\in\mathbb{C}^{-}$ ($\mathbb{C}^{-}$ is the lower part of the complex plane).

The total Hamiltonian $H=H_S+H_F+H_{\rm int}$ can be diagonalized in terms of the creation (annihilation) operators $A^{\dagger}_{\mathbf{k}}$ ($A_{\mathbf{k}}$):

$$H=\int d\mathbf{k}\,\omega_k A^{\dagger}_{\mathbf{k}}A_{\mathbf{k}},\quad [A_{\mathbf{k}},A^{\dagger}_{\mathbf{k}'}]=\delta^3(\mathbf{k}-\mathbf{k}'), \tag{5}$$

where

$$A^{\dagger}_{\mathbf{k}}\equiv a^{\dagger}_{\mathbf{k}}+\frac{V_k}{\eta_{+}(\omega_k)}\left[b^{\dagger}+\int\frac{d\mathbf{k}'V_{k'}a^{\dagger}_{\mathbf{k}'}}{\omega_k-\omega_{k'}+i0}\right],$$

$$A_{\mathbf{k}}\equiv a_{\mathbf{k}}+\frac{V_k}{\eta_{-}(\omega_k)}\left[b+\int\frac{d\mathbf{k}'V_{k'}a_{\mathbf{k}'}}{\omega_k-\omega_{k'}-i0}\right]. \tag{6}$$

The operators $b^{\dagger}$, b, $a^{\dagger}_{\mathbf{k}}$, and $a_{\mathbf{k}}$ can be written in terms of $A^{\dagger}_{\mathbf{k}}$ and $A_{\mathbf{k}}$:

$$b^{\dagger}=\int\frac{d\mathbf{k}V_k}{\eta_{-}(\omega_k)}A^{\dagger}_{\mathbf{k}},$$

$$b=\int\frac{d\mathbf{k}V_k}{\eta_{+}(\omega_k)}A_{\mathbf{k}},$$

$$a^{\dagger}_{\mathbf{p}}=\int d\mathbf{k}\left[\delta^3(\mathbf{k}-\mathbf{p})+\frac{V_kV_p}{\eta_{-}(\omega_k)(\omega_k-\omega_p-i0)}\right]A^{\dagger}_{\mathbf{k}},$$

$$a_{\mathbf{p}}=\int d\mathbf{k}\left[\delta^3(\mathbf{k}-\mathbf{p})+\frac{V_kV_p}{\eta_{+}(\omega_k)(\omega_k-\omega_p+i0)}\right]A_{\mathbf{k}}. \tag{7}$$

Equations (5) and (7) can be proved using Eqs. (1)–(4) and the following identities:

$$\frac{1}{\eta_{\pm}(\omega_k)}=-\int\frac{d\mathbf{k}\;V_k^2}{\eta_{+}(\omega_k)\,\eta_{-}(\omega_k)(\omega_k-\omega_p\mp i0)}, \tag{8}$$

$$\int\frac{d\mathbf{k}\;V_k^2}{\eta_{+}(\omega_k)\,\eta_{-}(\omega_k)}=1, \tag{9}$$

$$\frac{1}{\eta_{-}(\omega_{k'})(\omega_{k'}-\omega_{k''}-i0)}+\frac{1}{\eta_{+}(\omega_{k''})(\omega_{k''}-\omega_{k'}+i0)}$$
$$=-\int\frac{d\mathbf{k}\;V_k^2}{\eta_{+}(\omega_k)\,\eta_{-}(\omega_k)(\omega_k-\omega_{k''}-i0)(\omega_k-\omega_{k'}+i0)}. \tag{10}$$

Equations (8), (9), and (10) are proved in Appendix A.

III. STATES AND OBSERVABLES

As is discussed in Refs. [17–19], for quantum systems with continuous spectrum the usual approach of density operators is not applicable. For a density operator $\hat{\rho}$ representing a state and an operator $\hat{O}$ representing an observable, expressions such as $\mathrm{Tr}(\hat{\rho}\hat{O})$ are meaningless due to the presence of singular diagonal terms. The usual way to avoid these problems is to "discretize" the spectrum by enclosing the system in a box with periodic boundary conditions. The size of the box is considered to be infinite only at the last step of the computation of relevant quantities.

A way to consider the continuous spectrum from the beginning was introduced in Refs. [17–19]. For a given set $\mathcal{O}$ of operators representing physical observables, the states of the system can be represented by a set $\mathcal{S}$ of functionals acting on $\mathcal{O}$ ($\mathcal{S}\subset O^{\times}$). The mean value of an observable $O\in\mathcal{O}$ in a state $\rho\in\mathcal{S}$ is given by the value of the functional ρ on O, denoted by $\langle O\rangle_{\rho}=(\rho|O)$. This expression generalizes the usual $\mathrm{Tr}(\hat{\rho}\hat{O})$. As usual, the time evolution of the observables in the Heisenberg representation is given by

$$O_t=e^{i\mathbb{L}^{\dagger}t}O, \quad \text{where } \mathbb{L}^{\dagger}O\equiv[H,O], \quad O\in\mathcal{O}. \tag{11}$$

For the model presented in Sec. II, let us consider a special set $\mathcal{O}$ of observables given by a generalized linear combination of products of one creation and one annihilation operator:

$$O=O_1 b^{\dagger}b+\int d\mathbf{k}\int d\mathbf{k}'\,\widetilde{O}_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'}+\int d\mathbf{k}O_{\mathbf{k}1}a_{\mathbf{k}}^{\dagger}b+\int d\mathbf{k}'\,O_{1\mathbf{k}'}b^{\dagger}a_{\mathbf{k}'}, \tag{12}$$

$$O_1^{*}=O_1, \quad \widetilde{O}_{\mathbf{k}\mathbf{k}'}^{*}=\widetilde{O}_{\mathbf{k}'\mathbf{k}}, \quad O_{\mathbf{k}1}^{*}=O_{1\mathbf{k}}.$$

Due to Eq. (11) and the form of the total Hamiltonian $H=H_S+H_F+H_{\mathrm{int}}$ given by Eqs. (1), (2), and (3), this form is preserved by time evolution ($e^{i\mathbb{L}^{\dagger}t}\mathcal{O}\subset\mathcal{O}$).

Some observables in $\mathcal{O}$ are especially important for our purposes:

$$n\equiv b^{\dagger}b \quad \text{number of discrete modes,}$$

$$N\equiv\int d\mathbf{k}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} \quad \text{number of continuous modes,} \tag{13}$$

$$\sigma(\mathbf{r})\equiv\psi^{\dagger}(\mathbf{r})\psi(\mathbf{r})$$

$$\text{density of continuous modes at the point } \mathbf{r},$$

where in the last expression $\psi^{\dagger}(\mathbf{r})\equiv(1/\sqrt{8\pi^3})\int d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}^{\dagger}$ is the creation operator for one continuous mode at the position $\mathbf{r}$. Notice that to include observables such as N or H in the set $\mathcal{O}$ it is necessary to allow $\widetilde{O}_{\mathbf{k}\mathbf{k}'}$ to have a singular part proportional to $\delta^3(\mathbf{k}-\mathbf{k}')$.

For the cases in which the total number of excited modes is finite ($\langle N_T\rangle=\langle n\rangle+\langle N\rangle<\infty$), observables such as H and N are well defined and should be included in $\mathcal{O}$. Therefore, in this case we denote by $\mathcal{O}_D$ (the D corresponds to ''decaying'' processes) the set of observables of the form (12), for which

$$\widetilde{O}_{\mathbf{k}\mathbf{k}'}=O_{\mathbf{k}}\delta^3(\mathbf{k}-\mathbf{k}')+O_{\mathbf{k}\mathbf{k}'}, \tag{14}$$

where $O_{\mathbf{k}}$, $O_{\mathbf{k}\mathbf{k}'}$, $O_{\mathbf{k}1}$, and $O_{1\mathbf{k}'}$ are regular functions of the variables $\mathbf{k}$ and $\mathbf{k}'$.

For all $O\in\mathcal{O}_D$ we obtain

$$\langle O\rangle_{\rho}=(\rho|O)=\rho_1^{*}O_1+\int d\mathbf{k}\,\rho_{\mathbf{k}}^{*}O_{\mathbf{k}}+\int\int d\mathbf{k}\,d\mathbf{k}'\rho_{\mathbf{k}\mathbf{k}'}^{*}O_{\mathbf{k}\mathbf{k}'}+\int d\mathbf{k}\,\rho_{\mathbf{k}1}^{*}O_{\mathbf{k}1}+\int d\mathbf{k}'\,\rho_{1\mathbf{k}'}^{*}O_{1\mathbf{k}'}, \tag{15}$$

where

$$\rho_1^{*}\equiv(\rho|b^{\dagger}b), \quad \rho_{\mathbf{k}}^{*}\equiv(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}), \quad \rho_{\mathbf{k}\mathbf{k}'}^{*}\equiv(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'}),$$
$$\rho_{\mathbf{k}1}^{*}\equiv(\rho|a_{\mathbf{k}}^{\dagger}b), \quad \rho_{1\mathbf{k}'}^{*}\equiv(\rho|b^{\dagger}a_{\mathbf{k}'}). \tag{16}$$

Therefore, the states $\rho\in\mathcal{S}\subset\mathcal{O}_D^{\times}$ are represented by the ''components'' ρ_1^{*}, $\rho_{\mathbf{k}}^{*}$, $\rho_{\mathbf{k}\mathbf{k}'}^{*}$, $\rho_{\mathbf{k}1}^{*}$, and $\rho_{1\mathbf{k}'}^{*}$, while the observables $O\in\mathcal{O}_D$ are represented by the ''components'' O_1, $O_{\mathbf{k}}$, $O_{\mathbf{k}\mathbf{k}'}$, $O_{\mathbf{k}1}$, and $O_{1\mathbf{k}'}$. We stress the need for ''singular components'' $\rho_{\mathbf{k}}^{*}$ and $O_{\mathbf{k}}$ in both states and observables.

The condition on the states representing a finite number of excited modes is given by

$$\rho_1^{*}+\int d\mathbf{k}\rho_{\mathbf{k}}^{*}=\langle n+N\rangle_{\rho}<\infty. \tag{17}$$

As the total number of modes operator $N_T\equiv n+N$ commutes with the total Hamiltonian, the value of $\langle n+N\rangle_{\rho}$ is time independent. ρ_1^{*} gives the mean number of excited modes for the oscillator, while $\rho_{\mathbf{k}}^{*}d\mathbf{k}$ is the mean number of continuous modes having momentum between $\mathbf{k}$ and $\mathbf{k}+d\mathbf{k}$.

If we wish to describe a situation in which there is an infinite number of excited modes in the field ($\langle N\rangle=\infty$), as is the case when the oscillator is interacting with a bath of radiation with uniform concentration of continuous modes, we need a different characterization of observables and states. In this case, extensive observables such as N or H are not well defined and should be excluded from the set of observables $\mathcal{O}$. Therefore, only intensive observables of the field are accessible for measurement, and we define the class $\mathcal{O}_{\mathrm{TD}}$ (the TD corresponds to ''thermodynamic limit'') of observables of the form (12), for which

$$\widetilde{O}_{\mathbf{k}\mathbf{k}'}=O_{\mathbf{k}\mathbf{k}'}, \tag{18}$$

where $O_{\mathbf{k}\mathbf{k}'}$, $O_{\mathbf{k}1}$, and $O_{1\mathbf{k}'}$ are regular functions of the variables $\mathbf{k}$ and $\mathbf{k}'$. For the mean value of any observable $O\in\mathcal{O}_{\mathrm{TD}}$ we have

$$\langle O\rangle_{\rho}=(\rho|O)=\rho_1^{*}O_1+\int\int d\mathbf{k}\,d\mathbf{k}'\,\widetilde{\rho}_{\mathbf{k}\mathbf{k}'}^{*}O_{\mathbf{k}\mathbf{k}'}+\int d\mathbf{k}\,\rho_{\mathbf{k}1}^{*}O_{\mathbf{k}1}+\int d\mathbf{k}'\,\rho_{1\mathbf{k}'}^{*}O_{1\mathbf{k}'}, \tag{19}$$

where $\rho_1^{*}\equiv(\rho|b^{\dagger}b)$, $\widetilde{\rho}_{\mathbf{k}\mathbf{k}'}^{*}\equiv(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'})$, $\rho_{\mathbf{k}1}^{*}\equiv(\rho|a_{\mathbf{k}}^{\dagger}b)$, and $\rho_{1\mathbf{k}'}^{*}\equiv(\rho|b^{\dagger}a_{\mathbf{k}'})$.

A singular part should be included in $\widetilde{\rho}_{\mathbf{k}\mathbf{k}'}^{*}$, if we want to consider the possibility of states having a uniform concentration of continuous modes. This fact can be understood by computing the mean value of the density given by Eq. (13):

$$\langle \sigma(\mathbf{r})\rangle = \frac{1}{8\pi^3}\int d\mathbf{k}\, d\mathbf{k}' e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}}\tilde{\rho}^*_{\mathbf{kk}'} .$$

For a state with uniform concentration c we should have $\tilde{\rho}^*_{\mathbf{kk}'}$ proportional to $\delta^3(\mathbf{k}-\mathbf{k}')$. Therefore we assume the general form

$$\tilde{\rho}^*_{\mathbf{kk}'} = \rho^*_{\mathbf{k}}\delta^3(\mathbf{k}-\mathbf{k}') + \rho^*_{\mathbf{kk}'} , \tag{20}$$

where $\rho^*_{\mathbf{k}}$ and $\rho^*_{\mathbf{kk}'}$ are regular functions of the variables $\mathbf{k}$ and $\mathbf{k}'$. Combining Eqs. (19) and (20), we obtain the general expression

$$\langle O\rangle_\rho = (\rho|O) = \rho^*_1 O_1 + \int d\mathbf{k}\ \rho^*_{\mathbf{k}} O_{\mathbf{kk}} + \int\int d\mathbf{k}\ d\mathbf{k}' \rho^*_{\mathbf{kk}'} O_{\mathbf{kk}'} + \int d\mathbf{k}\ \rho^*_{\mathbf{k}1} O_{\mathbf{k}1} + \int d\mathbf{k}'\ \rho^*_{1\mathbf{k}'} O_{1\mathbf{k}'}, \tag{21}$$

for computing the mean value of any observable $O \in \mathcal{O}_{\text{TD}}$ in a state $\rho \in \mathcal{S} \subset \mathcal{O}^{\times}_{\text{TD}}$. The states ρ are represented by their ''components'' ρ^*_1, $\rho^*_{\mathbf{k}}$, $\rho^*_{\mathbf{kk}'}$, $\boldsymbol{\rho}^*_{\mathbf{k}1}$, and $\rho^*_{1\mathbf{k}'}$, where in general $\rho^*_{\mathbf{k}}$ and $\rho^*_{\mathbf{kk}'}$ are independent objects ($\rho^*_{\mathbf{k}} \neq \rho^*_{\mathbf{kk}}$), while O is represented by the ''components'' O_1, $O_{\mathbf{kk}'}$, $O_{\mathbf{k}1}$, and $O_{1\mathbf{k}'}$. Notice that in this case there is no singular part for the observables.

IV. TIME EVOLUTION

In the Heisenberg picture, the functionals ρ representing states are time independent, while the observables evolve in time according to

$$-i\frac{d}{dt}O = \mathbb{L}^\dagger O \equiv [H,O].$$

For the creation and annihilation operators $A^\dagger_{\mathbf{k}}$ and $A_{\mathbf{k}}$ given in Eq. (6), we obtain

$$\frac{d}{dt}A^\dagger_{\mathbf{k}} = i\omega_k A^\dagger_{\mathbf{k}}, \quad \frac{d}{dt}A_{\mathbf{k}} = -i\omega_k A_{\mathbf{k}},$$

and therefore

$$A^\dagger_{\mathbf{k}}(t) = e^{i\omega_k t}A^\dagger_{\mathbf{k}}(0), \quad A_{\mathbf{k}}(t) = e^{-i\omega_k t}A_{\mathbf{k}}(0). \tag{22}$$

The time evolution of $a^\dagger_{\mathbf{k}}$, $a_{\mathbf{k}}$, $b^\dagger$, and b can be easily obtained replacing Eq. (22) in Eq. (7):

$$b^\dagger(t) = \int \frac{d\mathbf{k}\ V_k}{\eta_-(\omega_k)} e^{i\omega_k t}A^\dagger_{\mathbf{k}}(0),$$

$$b(t) = \int \frac{d\mathbf{k}\ V_k}{\eta_+(\omega_k)} e^{-i\omega_k t}A_{\mathbf{k}}(0), \tag{23}$$

$$a^\dagger_{\mathbf{p}}(t) = e^{i\omega_p t}A^\dagger_{\mathbf{p}}(0) + \int \frac{d\mathbf{k}\ V_k V_p}{\eta_-(\omega_k)(\omega_k-\omega_p-i0)} e^{i\omega_k t}A^\dagger_{\mathbf{k}}(0),$$

$$a_{\mathbf{p}}(t) = e^{-i\omega_p t}A_{\mathbf{p}}(0) + \int \frac{d\mathbf{k}\ V_k V_p}{\eta_+(\omega_k)(\omega_k-\omega_p+i0)} e^{-i\omega_k t}A_{\mathbf{k}}(0). \tag{24}$$

Using Eqs. (23) and (24) we obtain

$$(\rho|b^\dagger b)_t = \int \frac{d\mathbf{k}d\mathbf{k}'\, V_k V_{k'} (\rho|A^\dagger_{\mathbf{k}}A_{\mathbf{k}'})_{t=0} e^{i(\omega_k-\omega_{k'})t}}{\eta_-(\omega_k)\,\eta_+(\omega_{k'})} \tag{25}$$

$$\begin{aligned}(\rho|a^\dagger_{\mathbf{p}}a_{\mathbf{p}'})_t &= (\rho|A^\dagger_{\mathbf{p}}A_{\mathbf{p}'})_{t=0} e^{i(\omega_p-\omega_{p'})t} + \int \frac{d\mathbf{k}'\, V_{k'}V_{p'}(\rho|A^\dagger_{\mathbf{p}}A_{\mathbf{k}'})_{t=0} e^{i(\omega_p-\omega_{k'})t}}{\eta_+(\omega_{k'})(\omega_{k'}-\omega_{p'}+i0)} \\ &+ \int \frac{d\mathbf{k}\ V_k V_p (\rho|A^\dagger_{\mathbf{k}}A_{\mathbf{p}'})_{t=0} e^{i(\omega_k-\omega_{p'})t}}{\eta_-(\omega_k)(\omega_k-\omega_p-i0)} \\ &+ \int \frac{d\mathbf{k}d\mathbf{k}'\, V_k V_{k'} (\rho|A^\dagger_{\mathbf{k}}A_{\mathbf{k}'})_{t=0} e^{i(\omega_k-\omega_{k'})t}}{\eta_+(\omega_{k'})(\omega_{k'}-\omega_{p'}+i0)\,\eta_-(\omega_k)(\omega_k-\omega_p-i0)}.\end{aligned} \tag{26}$$

In the previous expressions $A^\dagger_{\mathbf{k}}(0)$ and $A_{\mathbf{k}}(0)$ can be expressed in terms of $a^\dagger_{\mathbf{k}}(0)$, $a_{\mathbf{k}}(0)$, $b^\dagger(0)$, and $b(0)$ using Eq. (6).

The total number of modes, given by the operator

$$N_T \equiv n + N = b^\dagger b + \int d\mathbf{k} a^\dagger_{\mathbf{k}} a_{\mathbf{k}},$$

satisfies

$$[H, N_T] = 0 \tag{27}$$

and therefore N_T is conserved during time evolution. Two different physical situations will be discussed in the following subsections: first we will analyze the case with a finite number of modes, and then the case with an infinite number of modes (thermodynamic limit).

A. Finite number of excited modes (decaying process)

In this case, extensive observables of the field such as the number of modes $\langle N\rangle$ or the energy $\langle H_F\rangle$ are finite. In Eqs. (12), (14), (15), and (16) we obtained the general form for states and observables.

Let us consider an initial state for which the mean number of excited modes in the oscillators is $\langle n\rangle_0$, and the mean number of modes of the field is $\langle N\rangle_0$, with a momentum distribution $f(\mathbf{k})$, i.e.,

$$\rho^*_1 = \langle b^\dagger b\rangle_0 = \langle n\rangle_0, \quad \rho^*_{\mathbf{k}} = \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle_0 = f(\mathbf{k}),$$

$$\int d\mathbf{k} f(\mathbf{k}) = \langle N\rangle_0, \quad \rho^*_{\mathbf{kk}'} = \rho^*_{\mathbf{k}1} = \rho^*_{1\mathbf{k}'} = 0. \tag{28}$$

The density $\sigma(\mathbf{r})$ of continuous modes at the point $\mathbf{r}$, defined in Eq. (13), is of the form given in Eqs. (12) and (14), with

$$[\sigma(\mathbf{r})]_1=[\sigma(\mathbf{r})]_{\mathbf{k}}=[\sigma(\mathbf{r})]_{\mathbf{k}1}=[\sigma(\mathbf{r})]_{1\mathbf{k}}=0,$$

$$[\sigma(\mathbf{r})]_{\mathbf{k}\mathbf{k}'}=\frac{1}{8\pi^3}e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}}.$$

The mean value of the density $\sigma(\mathbf{r})$ in the state defined by Eq. (28) is obtained using Eq. (15):

$$\langle\sigma(\mathbf{r})\rangle_\rho=(\rho|\sigma(\mathbf{r}))=0,$$

which is a reasonable result, as we are dealing with a finite number $\langle N\rangle_0$ of continuous modes in an infinite volume $(\mathbb{R}^3)$.

From Eqs. (6) and (28) we obtain

$$(\rho|A^\dagger_{\mathbf{p}}A_{\mathbf{p}})_0=f(\mathbf{p})+\frac{V_p^2\langle n\rangle_0}{\eta_+(\omega_p)\,\eta_-(\omega_p)}, \tag{29}$$

$$(\rho|A^\dagger_{\mathbf{p}}A_{\mathbf{p}'})_0=\frac{V_pV_{p'}\langle n\rangle_0}{\eta_+(\omega_p)\,\eta_-(\omega_{p'})}. \tag{30}$$

Replacing Eqs. (30) in (25) we obtain

$$\langle b^\dagger b\rangle_t=\langle n\rangle_0\left|\int\frac{d\mathbf{k}\;V_k^2e^{i\omega_kt}}{\eta_+(\omega_k)\,\eta_-(\omega_k)}\right|^2. \tag{31}$$

Replacing Eqs. (29) and (30) in Eq. (26) (with $\mathbf{p}=\mathbf{p}'$), we obtain

$$\langle a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\rangle_t=f(\mathbf{p})+\frac{\langle n\rangle_0V_p^2}{\eta_+(\omega_p)\,\eta_-(\omega_p)}+\frac{\langle n\rangle_0V_p^2e^{i\omega_pt}}{\eta_+(\omega_p)}\int\frac{d\mathbf{k}'\,V_{k'}^2e^{-i\omega_{k'}t}}{\eta_+(\omega_{k'})\,\eta_-(\omega_{k'})(\omega_{k'}-\omega_p+i0)}$$
$$+\frac{\langle n\rangle_0V_p^2e^{-i\omega_pt}}{\eta_-(\omega_p)}\int\frac{d\mathbf{k}\;V_k^2e^{i\omega_kt}}{\eta_+(\omega_k)\,\eta_-(\omega_k)(\omega_k-\omega_p-i0)}+\int\int\frac{d\mathbf{k}d\mathbf{k}'\,V_{k'}^2V_k^2\langle n\rangle_0V_p^2e^{i(\omega_k-\omega_{k'})t}}{|\eta_+(\omega_k)|^2|\eta_+(\omega_{k'})|^2(\omega_k-\omega_p-i0)(\omega_{k'}-\omega_p+i0)}. \tag{32}$$

The Riemann-Lebesgue theorem can be used in Eqs. (31) and (32) to obtain

$$\lim_{t\to\infty}\langle b^\dagger b\rangle_t=\lim_{t\to\infty}\langle n\rangle_t=0,$$

$$\lim_{t\to\infty}\langle a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\rangle_t=f(\mathbf{p})+\frac{\langle n\rangle_0V_p^2}{\eta_+(\omega_p)\,\eta_-(\omega_p)}. \tag{33}$$

From Eqs. (9), (13), and the previous equation we obtain

$$\lim_{t\to\infty}\langle N\rangle_t=\lim_{t\to\infty}\int d\mathbf{p}\langle a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\rangle_t$$
$$=\int d\mathbf{p}f(\mathbf{p})+\langle n\rangle_0\int\frac{d\mathbf{p}\;V_p^2}{\eta_+(\omega_p)\,\eta_-(\omega_p)}$$
$$=\langle N\rangle_0+\langle n\rangle_0.$$

Therefore the discrete system decays to the vacuum, transferring the initial number of modes to the field [as we pointed out in Eq. (27), the total number of modes is a constant of motion].

The momentum distribution for $t\to\infty$ is equal to the initial distribution $f(\mathbf{p})$ plus an isotropic term having a sharp peak around $\omega_p=\Omega$ (it is easy to prove that for small interactions $[V_p^2/\eta_+(\omega_p)\,\eta_-(\omega_p)]\simeq\delta(\omega_p-\Omega)/4\pi\Omega^2$). The energy of the discrete system is completely transferred to the field.

Expression (31) is a well known result for Friedrichs model [12,21], which we reobtain in the functional approach. From this expression it is possible to prove that $d/dt\langle b^\dagger b\rangle_{t=0}=0$ (Zeno regime) and also that for $t\to\infty$, the asymptotic form of $\langle b^\dagger b\rangle_t$ is proportional to t^{-6} (Khalfin regime [22] corresponding to $\omega_k=|\mathbf{k}|$, see Appendix C for the proof). Therefore, the functional approach applied to the model with a finite number of excited modes does not allow exponential decay.

B. Infinite number of excited modes (thermodynamic limit)

In this case extensive observables such as energy or number of modes of the field are not well defined, since they are really infinite and therefore cannot be considered. Only local observables of the field such as the density $\sigma(\mathbf{r})$ defined in Eq. (13) are available. In Eqs. (18)–(21) we obtained the general form of observables, states, and mean values.

We are going to consider the initial condition for which the mean number of discrete modes in the system is $\langle n\rangle_0<\infty$:

$$(\rho|b^\dagger b)_{t=0}=\langle n\rangle_0, \tag{34}$$

and the field has a uniform distribution in the space with concentration c:

$$(\rho|\sigma(\mathbf{r}))=\frac{1}{8\pi^3}\int d\mathbf{k}\int d\mathbf{k}'\,e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}}(\rho|a^\dagger_{\mathbf{k}}a_{\mathbf{k}'})=c.$$

From the previous equation we obtain

$$\tilde{\rho}^{*}_{\mathbf{kk}'}=(\rho|a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}'})_{t=0}=\rho(\mathbf{k})\,\delta^{3}(\mathbf{k}-\mathbf{k}'),\qquad \int d\mathbf{k}\rho(\mathbf{k})=8\pi^{3}c. \tag{35}$$

We also assume for simplicity that there is no initial correlation between the discrete system and the field:

$$\rho^{*}_{\mathbf{k}1}=(\rho|a^{\dagger}_{\mathbf{k}}b)_{t=0}=0,\quad \rho^{*}_{1\mathbf{k}}=(\rho|b^{\dagger}a_{\mathbf{k}})_{t=0}=0. \tag{36}$$

Using Eqs. (34), (35), (36), and the definition (6) for $A^{\dagger}_{\mathbf{k}}$ we obtain

$$\begin{aligned}(\rho|A^{\dagger}_{\mathbf{p}}A_{\mathbf{p}'})_{t=0}=&\rho(\mathbf{p})\,\delta^{3}(\mathbf{p}-\mathbf{p}')+\frac{\langle n\rangle_0 V_pV_{p'}}{\eta_{+}(\omega_p)\,\eta_{-}(\omega_{p'})}\\&+\frac{V_pV_{p'}\rho(\mathbf{p})}{\eta_{-}(\omega_{p'})(\omega_{p'}-\omega_p-i0)}\\&+\frac{V_pV_{p'}\rho(\mathbf{p}')}{\eta_{+}(\omega_{p})(\omega_{p}-\omega_{p'}+i0)}\\&+\frac{V_pV_{p'}}{\eta_{+}(\omega_p)\,\eta_{-}(\omega_{p'})}\\&\times\int\frac{d\mathbf{k}\;V_k^2\rho(\mathbf{k})}{(\omega_p-\omega_k+i0)(\omega_{p'}-\omega_k-i0)}.\end{aligned} \tag{37}$$

The last expression can be replaced in Eqs. (25) and (26) to obtain the explicit expressions for $\langle b^{\dagger}b\rangle_t$ and $\langle a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}'}\rangle_t$. The time dependence is rather complicated, but the Riemann-Lebesgue theorem can be used to eliminate the oscillating terms and to obtain the following asymptotic expressions for $t\to\infty$:

$$\lim_{t\to\infty}(\rho|b^{\dagger}b)_t=\int\frac{d\mathbf{p}\;V_p^2\rho(\mathbf{p})}{\eta_{+}(\omega_p)\,\eta_{-}(\omega_p)}, \tag{38}$$

$$\begin{aligned}\operatorname{W}\lim_{t\to\infty}&(\rho|a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}'})_t\\=&\rho(\mathbf{p})\,\delta^{3}(\mathbf{p}-\mathbf{p}')+\frac{V_pV_{p'}\rho(\mathbf{p}')}{\eta_{-}(\omega_{p'})(\omega_{p'}-\omega_p-i0)}\\&+\frac{V_pV_{p'}\rho(\mathbf{p})}{\eta_{+}(\omega_p)(\omega_p-\omega_{p'}+i0)}\\&+\int\frac{d\mathbf{k}\;V_k^2\rho(\mathbf{k})V_pV_{p'}}{\eta_{-}(\omega_k)\,\eta_{+}(\omega_k)(\omega_k-\omega_p-i0)(\omega_k-\omega_{p'}+i0)}.\end{aligned} \tag{39}$$

The last limit should be understood in the weak sense, i.e.,

$$\lim_{t\to\infty}\int\int d\mathbf{p}\;d\mathbf{p}'(\rho|a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}'})_tO_{\mathbf{pp}'}=\int\int d\mathbf{p}\;d\mathbf{p}'[\operatorname{W}\lim_{t\to\infty}(\rho|a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}'})_t]O_{\mathbf{pp}'}.$$

The main difference with the results of Sec. IV A is that in this case Eq. (38) shows that the discrete system does not decay towards the vacuum. The infinite number of modes of the field surrounding the discrete system produces a final state of the system independent of $\langle n\rangle_0$, but dependent on the interaction and the initial momentum distribution of the field.

If the initial condition for the bosonic field is a canonical distribution with temperature $T=1/\kappa\beta$ (κ is the Boltzmann constant), we have

$$\rho(\mathbf{k})=\frac{1}{e^{\beta\omega_k}-1}. \tag{40}$$

If, in addition, we assume a small interaction, we have

$$\frac{V_p^2}{\eta_{+}(\omega_p)\,\eta_{-}(\omega_p)}\simeq\frac{\delta(\omega_p-\Omega)}{4\pi\Omega^2}. \tag{41}$$

Replacing Eqs. (40) and (41) in Eq. (38) we obtain

$$\langle b^{\dagger}b\rangle_{t\to\infty}\simeq\frac{1}{e^{\beta\Omega}-1},$$

which is the mean number of modes of the discrete system we would have obtained for a quantum oscillator of frequency Ω in a canonical distribution with temperature $T=1/\kappa\beta$. This result cannot be obtained without the assumption of weak interaction.

V. THE MASTER EQUATION

In Sec. IV A we showed how the quantum oscillator decays toward the vacuum in the presence of a finite number of modes of the field. In Sec. IV B we showed how a thermal bath thermalizes the quantum oscillator. These results have been obtained without approximations using the Riemann-Lebesgue theorem only. It is instructive to compare these results with the predictions of the master equation obtained in the so-called "λ^2t approximation," where the interaction parameter $\lambda\to0$, $t\to\infty$, and λ^2t is finite [2].

The master equation provides the time evolution of $\mathbb{P}\rho$, where $\mathbb{P}$ is the projector onto the part of ρ that is invariant by the time evolution without interaction. (In this section we consider the Schrödinger representation, where the states are time dependent and the observables are time independent.) Calling by $\mathbb{L}_0$ and $\mathbb{L}=\mathbb{L}_0+\mathbb{L}_1$ the generators of free and interacting time evolution, we have

$$i\frac{d}{dt}\mathbb{P}\rho=\left(\mathbb{P}\mathbb{L}_1\mathbb{Q}\frac{1}{i0-\mathbb{L}_0}\mathbb{Q}\mathbb{L}_1\mathbb{P}\right)\mathbb{P}\rho,\quad \mathbb{P}^2=\mathbb{P},\quad \mathbb{Q}=\mathbb{I}-\mathbb{P}. \tag{42}$$

We consider the states ρ as functionals acting on the space $\mathcal{O}$ of observables ($\rho\in\mathcal{S}\subset\mathcal{O}^{\times}$). Therefore it is necessary to give a precise definition of the operators appearing in Eq. (42). For a given set $\mathcal{O}$ of observables, any superoperator $\mathbb{M}^{\dagger}$ acting on $\mathcal{O}$ for which $\mathbb{M}^{\dagger}\mathcal{O}\subset\mathcal{O}$ is extended to an operator $\mathbb{M}$ acting on the space of states $\mathcal{S}\subset\mathcal{O}^{\times}$ by the following definition:

$$(\mathbb{M}\rho|O)=(\rho|\mathbb{M}^{\dagger}O),\quad \text{for all } O\in\mathcal{O}. \tag{43}$$

Therefore we have

$$(\mathbb{L}_0\rho|O)=(\rho|\mathbb{L}_0^{\dagger}O),\quad (\mathbb{L}\rho|O)=(\rho|\mathbb{L}^{\dagger}O)$$

$$\mathbb{L}_0^{\dagger}O\equiv[H_0,O],\quad \mathbb{L}^{\dagger}O\equiv[H,O]. \tag{44}$$

For the model presented in Sec. II,

$$H_0=\Omega b^{\dagger}b+\int d\mathbf{k}\omega_k a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}},$$

$$V=\int d\mathbf{k}V_k[a_{\mathbf{k}}^{\dagger}b+a_{\mathbf{k}}b^{\dagger}],\quad H=H_0+V.$$

As we shall see, completely different physical processes can be described with the same formal Eq. (42) if we allow or not for a singular part in the set $\mathcal{O}$ of ''test observables.''

A. Master equation for the decay process

We consider the class of observables $\mathcal{O}\in\mathcal{O}_D$, as we defined in Eqs. (12), (14), (15), and (16). In this case an observable O is represented by the constant O_1 and the regular functions $O_{\mathbf{k}}$, $O_{\mathbf{kk}'}$, $O_{\mathbf{k}1}$, and $O_{1\mathbf{k}}$, while the state ρ is represented by the constant ρ_1^* and the regular functions $\rho_{\mathbf{k}}^*$, $\rho_{\mathbf{kk}'}^*$, $\rho_{\mathbf{k}1}^*$, and $\rho_{1\mathbf{k}}^*$.

The states $\mathbb{P}\rho$, invariant under the time evolution generated by $\mathbb{L}_0$, satisfy

$$\begin{aligned}0&=(\mathbb{L}_0\mathbb{P}\rho|O)=(\mathbb{P}\rho|\mathbb{L}_0^{\dagger}O)\\&=\int d\mathbf{k}\int d\mathbf{k}'(\mathbb{P}\rho)_{\mathbf{kk}'}^*(\omega_k-\omega_{k'})O_{\mathbf{kk}'}\\&\quad+\int d\mathbf{k}(\mathbb{P}\rho)_{\mathbf{k}1}^*(\omega_k-\Omega)O_{\mathbf{k}1}\\&\quad+\int d\mathbf{k}'(\mathbb{P}\rho)_{1\mathbf{k}'}^*(\Omega-\omega_{k'})O_{1\mathbf{k}'},\end{aligned} \tag{45}$$

for all $O\in\mathcal{O}_D$, which implies

$$(\mathbb{P}\rho)_{\mathbf{kk}'}^*=(\mathbb{P}\rho)_{\mathbf{k}1}^*=(\mathbb{P}\rho)_{1\mathbf{k}'}^*=0, \tag{46}$$

and therefore, using $(\mathbb{P}\rho|O)=(\rho|\mathbb{P}^{\dagger}O)$, we obtain

$$\mathbb{P}^{\dagger}O=O_1b^{\dagger}b+\int d\mathbf{k}O_{\mathbf{k}}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}. \tag{47}$$

The master equation (42) implies

$$\left(\frac{d}{dt}\mathbb{P}\rho\middle|\mathbb{P}^{\dagger}O\right)=\left[\mathbb{P}\rho\middle|\left(-i\mathbb{P}^{\dagger}\mathbb{L}^{\dagger}\mathbb{Q}^{\dagger}\frac{1}{\mathbb{L}_0^{\dagger}+i0}\mathbb{Q}^{\dagger}\mathbb{L}^{\dagger}\mathbb{P}^{\dagger}\right)\mathbb{P}^{\dagger}O\right]. \tag{48}$$

Two independent equations can be obtained if we choose the operators $b^{\dagger}b$ and $a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$ for $\mathbb{P}^{\dagger}O$. Using the definitions (44) and (47) for $\mathbb{L}_0^{\dagger}$, $\mathbb{L}^{\dagger}$, and $\mathbb{P}^{\dagger}$, we obtain

$$\frac{d}{dt}(\rho|b^{\dagger}b)=-8\pi^2\Omega^2V_{\Omega}^2(\rho|b^{\dagger}b), \tag{49}$$

$$\frac{d}{dt}(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}})=2\pi V_{\Omega}^2\delta(\omega_k-\Omega)(\rho|b^{\dagger}b). \tag{50}$$

The proof of these equations is given in Appendix B. The solution of Eqs. (49) and (50) is

$$(\rho|b^{\dagger}b)_t=e^{-\Gamma t}(\rho|b^{\dagger}b)_0,\quad \Gamma\equiv8\pi^2\Omega^2V_{\Omega}^2,$$

$$(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}})_t=[1-e^{-\Gamma t}]\frac{\delta(\omega_k-\Omega)}{4\pi\Omega^2}(\rho|b^{\dagger}b)_0+(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}})_0. \tag{51}$$

For $t\to\infty$, expressions given in Eqs. (51) give

$$(\rho|b^{\dagger}b)_{\infty}=0,$$

$$(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}})_{\infty}=\frac{\delta(\omega_k-\Omega)}{4\pi\Omega^2}(\rho|b^{\dagger}b)_0+(\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}})_0.$$

These expressions are coincident with the results given in Eq. (33) for small interaction. However, the master equation gives an exponential decay that is not found in the exact expressions (31) and (32). The reason is that for the master equation to be valid it is necessary that the interaction parameter λ be small ($\lambda\ll1$) and that the time be not too large ($t\lesssim\lambda^{-2}$). See Appendix D for a discussion on this point.

B. Master equation for the thermodynamic limit

We now consider the class of observables $\mathcal{O}_{\mathrm{TD}}$ given in Eqs. (18)–(21) corresponding to an infinite number of excited modes of the field.

States $\mathbb{P}\rho$ are invariant under the time evolution without interaction, and therefore they satisfy Eq. (45) for all $O\in\mathcal{O}_{\mathrm{TD}}$.

In addition, we assume that $\mathbb{P}\rho$ is translationally invariant for all the field observables, i.e.,

$$(\mathbb{L}_{\bar{P}}\mathbb{P}\rho|O_{\mathrm{field}})=(\mathbb{P}\rho|\mathbb{L}_{\bar{P}}^{\dagger}O_{\mathrm{field}})=(\mathbb{P}\rho|[\bar{P},O_{\mathrm{field}}])=0, \tag{52}$$

where

$$O_{\mathrm{field}}=\int d\mathbf{k}\int d\mathbf{k}'O_{\mathbf{kk}'}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'},\quad \bar{P}=\int d\mathbf{k}\mathbf{k}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}, \tag{53}$$

and $\mathbb{L}_{\bar{P}}$ is the generator of space translations for the states.

Equation (52) implies

$$\int d\mathbf{k}\int d\mathbf{k}'(\mathbb{P}\rho)_{\mathbf{kk}'}^*(\mathbf{k}-\mathbf{k}')O_{\mathbf{kk}'}=0,$$

from which we obtain

$$(\mathbb{P}\rho)_{\mathbf{kk}'}^*=(\mathbb{P}\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'})=\rho_{\mathbf{k}}^*\delta^3(\mathbf{k}-\mathbf{k}'). \tag{54}$$

We also assume there is no correlation between the field and the discrete system in $\mathbb{P}\rho$, i.e.,

$$(\mathbb{P}\rho|a_{\mathbf{k}}^{\dagger}b)=(\mathbb{P}\rho|b^{\dagger}a_{\mathbf{k}})=0. \tag{55}$$

In summary, $\mathbb{P}\rho$ and $\mathbb{Q}\rho$ satisfy

$$(\mathbb{P}\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'})=\rho_{\mathbf{k}}^{*}\delta^{3}(\mathbf{k}-\mathbf{k}'),\quad (\mathbb{Q}\rho|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'})=\rho_{\mathbf{kk}'}^{*},$$

$$(\mathbb{P}\rho|b^{\dagger}b)=\rho_{1}^{*},\quad (\mathbb{Q}\rho|b^{\dagger}b)=0,$$

$$(\mathbb{P}\rho|a_{\mathbf{k}}^{\dagger}b)=0,\quad (\mathbb{Q}\rho|a_{\mathbf{k}}^{\dagger}b)=\rho_{\mathbf{k}1}^{*},\tag{56}$$

$$(\mathbb{P}\rho|b^{\dagger}a_{\mathbf{k}})=0,\quad (\mathbb{Q}\rho|b^{\dagger}a_{\mathbf{k}})=\rho_{1\mathbf{k}}^{*},$$

and the mean value of $O\in\mathcal{O}_{\mathrm{TD}}$ in a general state $\rho=\mathbb{P}\rho+\mathbb{Q}\rho$ is

$$\langle O\rangle_{\rho}=(\rho|O)=\rho_{1}^{*}O_{1}+\int d\mathbf{k}\;\rho_{\mathbf{k}}^{*}O_{\mathbf{kk}}+\int d\mathbf{k}\int d\mathbf{k}'\,\rho_{\mathbf{kk}'}^{*}O_{\mathbf{kk}'}+\int d\mathbf{k}\;\rho_{\mathbf{k}1}^{*}O_{\mathbf{k}1}+\int d\mathbf{k}'\;\rho_{1\mathbf{k}'}^{*}O_{1\mathbf{k}'}.$$

The operator $\mathbb{P}^{\dagger}$ acting on $\mathcal{O}_{\mathrm{TD}}$ is defined by

$$(\rho|\mathbb{P}^{\dagger}O)\equiv(\mathbb{P}\rho|O)=\rho_{1}^{*}O_{1}+\int d\mathbf{k}\;\rho_{\mathbf{k}}^{*}O_{\mathbf{kk}}.$$

In Appendix B, the functional master equation (42) is evaluated on observables $O_{1}b^{\dagger}b$ and $\int d\mathbf{k}\int d\mathbf{k}'O_{\mathbf{kk}'}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'}$, obtaining the following differential equations:

$$\frac{d}{dt}\rho_{1}^{*}(t)=-8\pi\Omega^{2}V_{\Omega}^{2}\rho_{1}^{*}(t)+2\pi V_{\Omega}^{2}\int d\mathbf{k}\delta(\omega_{k}-\Omega)\rho_{\mathbf{k}}^{*}(t),\tag{57}$$

$$\frac{d}{dt}\rho_{\mathbf{k}}^{*}(t)=0.\tag{58}$$

The solution of these equations is

$$\rho_{1}^{*}(t)=\rho_{1}^{*}(0)e^{-\Gamma t}+\frac{[1-e^{-\Gamma t}]\int d\mathbf{k}\;\delta(\omega_{k}-\Omega)\rho_{\mathbf{k}}^{*}(0)}{4\pi\Omega^{2}},\tag{59}$$

$$\rho_{\mathbf{k}}^{*}(t)=\rho_{\mathbf{k}}^{*}(0).\tag{60}$$

Equation (60) states that the momentum distribution of the continuous modes does not change in time, while Eq. (59) gives the time evolution of the mean number of discrete modes. For $t\to\infty$ these equations give

$$\rho_{1}^{*}(\infty)=\frac{\int d\mathbf{k}\delta(\omega_{k}-\Omega)\rho_{\mathbf{k}}^{*}(0)}{4\pi\Omega^{2}},\quad \rho_{\mathbf{k}}^{*}(\infty)=\rho_{\mathbf{k}}^{*}(0),$$

which coincide with the exact expressions (38) and (39) for small interaction. Once again, the master equation gives an exponential approach that is not found in the exact time evolution (see Appendix D).

VI. CONCLUSIONS

Let us summarize our main results. We considered the linear coupling between a quantum oscillator and a quantum field with the Hamiltonian

$$H=\Omega b^{\dagger}b+\int d\mathbf{k}\omega_{k}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}+\int d\mathbf{k}\;V_{k}[a_{\mathbf{k}}^{\dagger}b+b^{\dagger}a_{\mathbf{k}}],$$

$$V_{k}^{*}=V_{k},\quad \hbar=c=1,$$

$$[b,b^{\dagger}]=1,\quad [a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}]=\delta^{3}(\mathbf{k}-\mathbf{k}'),\quad [a_{\mathbf{k}}^{\dagger},b^{\dagger}]=[a_{\mathbf{k}}^{\dagger},b]=0.$$

For a finite number of excited modes, we used the functional approach to obtain the time evolution of a state having a finite number of excited modes. The mean number of discrete modes is given by

$$\langle b^{\dagger}b\rangle_{t}=\langle n\rangle_{0}\left|\int\frac{d\mathbf{k}\;V_{k}^{2}e^{i\omega_{k}t}}{\eta_{+}(\omega_{k})\eta_{-}(\omega_{k})}\right|^{2},$$

$$\eta_{\pm}(\omega_{k})\equiv\omega_{k}-\Omega-\int\frac{d\mathbf{k}'V_{k'}^{2}}{\omega_{k}-\omega_{k'}\pm i0}.$$

This is a well known result [12,21], which we reobtain in the functional approach. From this expression it is possible to prove that $d/dt\langle b^{\dagger}b\rangle_{t=0}=0$ (Zeno regime) and also that for $t\to\infty$, the asymptotic form of $\langle b^{\dagger}b\rangle_{t}$ is proportional to t^{-6} (Khalfin regime [22] corresponding to $\omega_{\mathbf{k}}=|\mathbf{k}|$, see Appendix C for the proof). Therefore, the functional approach applied to the model with a finite number of excited modes does not allow exponential decay.

The functional approach is a powerful tool to deal with the case of an infinite number of excited modes in the field. In this case, the mean number of discrete modes approaches *in nonexponential form*

$$\langle b^{\dagger}b\rangle_{\infty}=\lim_{t\to\infty}(\rho|b^{\dagger}b)_{t}=\int\frac{d\mathbf{p}\;V_{p}^{2}\rho(\mathbf{p})}{\eta_{+}(\omega_{p})\eta_{-}(\omega_{p})}.$$

This number depends on the initial momentum distribution $\rho(\mathbf{p})$ of the field and the form of the interaction V_{p}, but it does not depend on the initial condition of the quantum oscillator. If the interaction is very small, $\langle b^{\dagger}b\rangle_{\infty}$ is independent of the form of the interaction, precisely

$$\langle b^{\dagger}b\rangle_{\infty}\simeq\frac{1}{4\pi\Omega^{2}}\int d\mathbf{p}\delta(\omega_{p}-\Omega)\rho(\mathbf{p}).$$

If in addition the initial momentum distribution of the field is the canonical distribution with temperature T, we obtained

$$\langle b^{\dagger}b\rangle_{\infty}\simeq\frac{1}{e^{\Omega/\kappa T}-1},$$

which is the mean number of modes for a single oscillator having temperature T. Therefore, for a small interaction, we proved the evolution of the oscillator to the thermal equilibrium with the field.

It is interesting to note the following.

(i) The space of ‘‘test observables’’ $\mathcal{O}_{D}$ for a finite number of excited modes is different from the space $\mathcal{O}_{\mathrm{TD}}$ for an infinite number of excited modes. In the latter case only intensive observables of the field are accessible for measurement, and therefore we excluded the possibility of a singular part $\int d\mathbf{k} O_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}$ from $\mathcal{O}_{\rm TD}$. This difference between $\mathcal{O}_D$ and $\mathcal{O}_{\rm TD}$ determines different time evolution for the states, since they are functionals whose properties depend on the domain of definition.

(ii) The strong limit of the state for infinite time does not exist, but a functional ρ_∞ exists such that $\lim_{t\to\infty}(\rho_t|O) = (\rho_\infty|O)$ for all O in the set of observables. We do not have, as is the case in the ''coarse graining'' method, a preferred set of relevant components of the state, obtained for a set of preferred observables. In our case we really consider the set of ''components'' $(\rho|O)$ of the states ρ, labeled by all the observables O. The difficulty of defining a canonical coarse graining is avoided in this approach.

(iii) As no analytic extensions have been involved to obtain the exact results, no special riggings of the spaces of states and observables like in Refs. [10], [11], [13], and [19] have been used. Only mild conditions on the momentum distribution of the field are necessary to use the Riemann-Lebesgue theorem in order to obtain the states for $t\to\infty$.

(iv) In Sec. V we used the functional approach to solve the master equation. Both for finite and infinite numbers of modes, the master equation gives a time evolution for which the $t\to\infty$ limit coincides with the corresponding limit of exact solutions with very small interactions. However, the master equation predicts exponential approaches that are not obtained in the exact solutions. These different results appear because, as we discuss in Appendix D, a necessary condition for the master equation to be valid is that the parameter λ of the interaction be small ($\lambda \ll 1$) *and* that the time be not too large ($t \lesssim \lambda^{-2}$). The master equation approximation eliminates the Khalfin effect.

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APPENDIX A: DIAGONALIZATION OF THE HAMILTONIAN

In this section we give the proof of Eqs. (8), (10), and (9), which are necessary to obtain the diagonalized expression (5) of the total Hamiltonian. The proof follows essentially the same arguments of Ref. [12], adapted to the model of Sec. II.

From

$$\eta_\pm(\omega_k) \equiv \omega_k - \Omega - \int \frac{d\mathbf{k}' V_{k'}^2}{\omega_k - \omega_{k'} \pm i0}, \qquad \omega_k = |\mathbf{k}|,$$

we obtain

$$\eta_+(\omega_k) - \eta_-(\omega_k) = 8\pi^2 i k^2 V_k^2$$

and

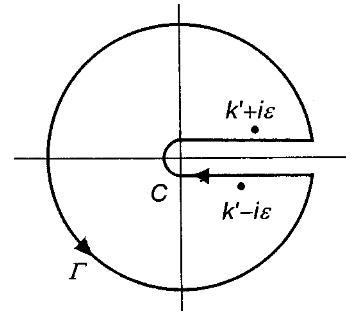

FIG. 1. Contour and poles for computing I_1 in Eq. (A3).

$$\frac{V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)} = \frac{1}{8\pi^2 i k^2}\left[\frac{1}{\eta_-(\omega_k)} - \frac{1}{\eta_+(\omega_k)}\right]. \quad \text{(A1)}$$

Therefore,

$$I_1 \equiv -\int \frac{d\mathbf{k} V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)(\omega_k - \omega_{k'} \mp i\varepsilon)}$$
$$= \int\int\int \frac{d\mathbf{k}}{8\pi^2 i k^2}\left[\frac{1}{\eta_+(\omega_k)} - \frac{1}{\eta_-(\omega_k)}\right]\frac{1}{(\omega_k - \omega_{k'} \mp i\varepsilon)}$$
$$= \frac{1}{2\pi i}\int_0^\infty \frac{dk}{k - k' - \mp i\varepsilon}\left[\frac{1}{\eta_+(k)} - \frac{1}{\eta_-(k)}\right]$$
$$= \frac{1}{2\pi i}\int_C \frac{dz}{\eta(z)(z - [k' \pm i\varepsilon])},$$

$$\eta(z) \equiv z - \Omega - \int \frac{d\mathbf{k}' V_{k'}^2}{z - \omega_{k'}}, \qquad \varepsilon > 0. \quad \text{(A2)}$$

In the last expression, C is the curve in the complex plane surrounding $\mathbb{R}^+$, as is shown in Fig. 1. The integrand behaves as z^{-2} when $|z|\to\infty$, and therefore the integral over C can be closed with a curve Γ as shown in Fig. 1. The integral over $C+\Gamma$ can be evaluated computing the residue at the point $k' \pm i\varepsilon$:

$$I_1 = \frac{1}{2\pi i}\int_{C+\Gamma} \frac{dz}{\eta(z)(z - [k' \pm i\varepsilon])} = \frac{1}{\eta(k' \pm i\varepsilon)}. \quad \text{(A3)}$$

Combining Eqs. (A2) and (A3) we obtain

$$\frac{1}{\eta_\pm(k')} = -\int \frac{d\mathbf{k}\, V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)(\omega_k - \omega_{k'} \mp i0)}, \quad \text{(A4)}$$

which is Eq. (8).

We also have

$$I_2 \equiv -\int \frac{d\mathbf{k}\, V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)(\omega_k - \omega_{k'} + i\varepsilon)(\omega_k - \omega_{k''} - i\varepsilon)}$$
$$= \frac{1}{2\pi i}\int_C \frac{dz}{\eta(z)(z - [k' - i\varepsilon])(z - [k'' + i\varepsilon])}, \quad \text{(A5)}$$

where the curve C is shown in Fig. 2. The integrand behaves as z^{-3} when $|z|\to\infty$, and once again the integral can be

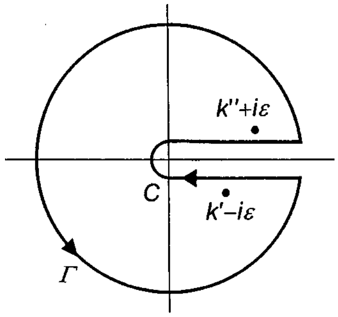

FIG. 2. Contour and poles for computing I_2 in Eq. (A6).

closed with the curve Γ shown in Fig. 2. The integral over $C+\Gamma$ can be evaluated computing the residue at the points $k'-i\varepsilon$ and $k''+i\varepsilon$:

$$I_2=\frac{1}{\eta_-(k')(k'-k''-i\varepsilon)}+\frac{1}{\eta_+(k'')(k''-k'+i\varepsilon)}. \tag{A6}$$

Combining Eqs. (A5) and (A6) we obtain Eq. (10):

$$-\int\frac{d\mathbf{k}\,V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)(\omega_k-\omega_{k'}+i0)(\omega_k-\omega_{k''}-i0)}$$
$$=\frac{1}{\eta_-(\omega_{k'})(\omega_{k'}-\omega_{k''}-i0)}+\frac{1}{\eta_+(\omega_{k''})(\omega_{k''}-\omega_{k'}+i0)}. \tag{A7}$$

Using Eq. (A1) we obtain

$$I_3\equiv\int\frac{d\mathbf{k}\,V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)}=\frac{1}{2\pi i}\int_0^\infty dk\left[\frac{1}{\eta(k-i0)}-\frac{1}{\eta(k+i0)}\right]$$
$$=-\frac{1}{2\pi i}\int_C\frac{dz}{\eta(z)}=\frac{1}{2\pi i}\int_\Gamma\frac{dz}{\eta(z)}, \tag{A8}$$

where the last identity follows from the analyticity of $1/\eta(z)$ in $\mathbb{C}-\mathbb{R}^+$, and the curves C and Γ are shown in Fig. 3. For $|z|\to\infty$ we have $\eta(z)\simeq z$ and therefore

$$\frac{1}{2\pi i}\int_\Gamma\frac{dz}{\eta(z)}=\lim_{|z|\to\infty}\frac{1}{2\pi i}\int_\Gamma\frac{dz}{z}. \tag{A9}$$

From Eqs. (A8) and (A9) we deduce Eq. (9):

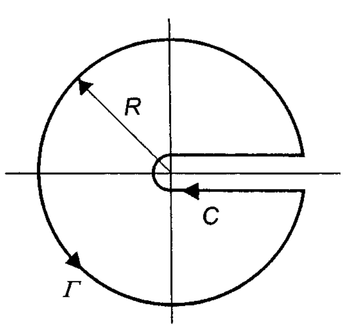

FIG. 3. Contour and poles for computing I_3 in Eq. (A8).

$$\int\frac{d\mathbf{k}\,V_k^2}{\eta_+(\omega_k)\,\eta_-(\omega_k)}=1. \tag{A10}$$

With Eqs. (A10), (A7), and (A4) the proof of Eqs. (5) and (7) is straightforward. Consider, for example,

$$\int\frac{d\mathbf{k}\,V_k}{\eta_-(\omega_k)}A_{\mathbf{k}}^\dagger=\int\frac{d\mathbf{k}\,V_k}{\eta_-(\omega_k)}a_{\mathbf{k}}^\dagger+b^\dagger\int\frac{d\mathbf{k}\,V_k^2}{\eta_-(\omega_k)\,\eta_+(\omega_k)}$$
$$+\int d\mathbf{k}'\,V_{k'}^2a_{\mathbf{k}'}^\dagger\int$$
$$\times\frac{d\mathbf{k}\,V_k}{\eta_-(\omega_k)\,\eta_+(\omega_k)(\omega_k-\omega_{k'}+i0)}.$$

Using Eqs. (A10) and (A4) we deduce

$$\int\frac{d\mathbf{k}\,V_k}{\eta_-(\omega_k)}A_{\mathbf{k}}^\dagger=b^\dagger,$$

which is the first equation in Eq. (7).

APPENDIX B: DEDUCTION OF THE MASTER EQUATION

Taking into account that $\mathbb{L}_V^\dagger O=[V,O]$ and $\mathbb{L}_0^\dagger O=[H_0,O]$, where

$$H_0=\Omega b^\dagger b+\int d\mathbf{k}\omega_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}},\quad V=\int d\mathbf{k}\,V_k[a_{\mathbf{k}}^\dagger b+b^\dagger a_{\mathbf{k}}], \tag{B1}$$

we obtain

$$\mathbb{L}_0^\dagger b^\dagger b=0, \tag{B2}$$

$$\mathbb{L}_0^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}=(\omega_k-\omega_{k'})a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}, \tag{B3}$$

$$\mathbb{L}_0^\dagger b^\dagger a_{\mathbf{k}}=(\Omega-\omega_k)b^\dagger a_{\mathbf{k}}, \tag{B4}$$

$$\mathbb{L}_0^\dagger a_{\mathbf{k}}^\dagger b=(\omega_k-\Omega)a_{\mathbf{k}}^\dagger b, \tag{B5}$$

and

$$\mathbb{L}_V^\dagger b^\dagger b=\int d\mathbf{k}\,V_k(a_{\mathbf{k}}^\dagger b-b^\dagger a_{\mathbf{k}}), \tag{B6}$$

$$\mathbb{L}_V^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}=V_k b^\dagger a_{\mathbf{k}'}-V_{k'}a_{\mathbf{k}}^\dagger b, \tag{B7}$$

$$\mathbb{L}_V^\dagger b^\dagger a_{\mathbf{k}}=\int d\mathbf{p}\,V_p a_{\mathbf{p}}^\dagger a_{\mathbf{k}}-V_k b^\dagger b, \tag{B8}$$

$$\mathbb{L}_V^\dagger a_{\mathbf{k}}^\dagger b=-\int d\mathbf{p}V_p a_{\mathbf{k}}^\dagger a_{\mathbf{p}}+V_k b^\dagger b. \tag{B9}$$

We first consider the deduction of the master equation for the decaying process. In this case, as we deduced in Sec. V A, the projector on the invariant part of the observables $\mathcal{O}_D$ under the time evolution generated by $\mathbb{L}_0$ is

$$\mathbb{P}^\dagger O=O_1 b^\dagger b+\int d\mathbf{k}O_{\mathbf{k}}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \tag{B10}$$

From Eqs. (B10), (B4), (B5), and (B6) we obtain

$$\frac{1}{\mathbb{L}_0+i0}\mathbb{Q}^\dagger\mathbb{L}^\dagger\mathbb{P}^\dagger b^\dagger b=\int d\mathbf{k}\ V_k\left(\frac{a_{\mathbf{k}}^\dagger b}{\omega_k-\Omega+i0}+\frac{b^\dagger a_{\mathbf{k}}}{\Omega-\omega_k+i0}\right). \tag{B11}$$

Using Eqs. (B8), (B9), and (B10),

$$\begin{aligned}\left(-i\mathbb{P}^\dagger\mathbb{L}^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}^\dagger\mathbb{P}^\dagger\right)b^\dagger &=i\int d\mathbf{k}\ V_k^2\left(\frac{1}{\Omega-\omega_k-i0}-\frac{1}{\Omega-\omega_k+i0}\right)b^\dagger b\\ &=-2\pi\int d\mathbf{k}\ V_k^2\delta(\Omega-\omega_k)b^\dagger b=-8\pi^2\Omega^2V_\Omega^2 b^\dagger b.\end{aligned} \tag{B12}$$

To obtain the last equation we used the relation

$$\left(\frac{1}{x-i0}-\frac{1}{x+i0}\right)=2\pi i\,\delta(x). \tag{B13}$$

Replacing Eq. (B12) in Eq. (48), with $O=b^\dagger b$ we obtain Eq. (49),

$$\frac{d}{dt}(\rho|b^\dagger b)=-8\pi^2\Omega^2V_\Omega^2(\rho|b^\dagger b).$$

From Eqs. (B10), (B7), (B4), and (B5),

$$\frac{1}{\mathbb{L}_0+i0}\mathbb{Q}^\dagger\mathbb{L}^\dagger\mathbb{P}^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{k}}=-V_k\left(\frac{a_{\mathbf{k}}^\dagger b}{\omega_k-\Omega+i0}-\frac{b^\dagger a_{\mathbf{k}}}{\Omega-\omega_k+i0}\right).$$

Using Eqs. (B10), (B8), and (B9),

$$\begin{aligned}\left(-i\mathbb{P}^\dagger\mathbb{L}^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}^\dagger\mathbb{P}^\dagger\right)a_{\mathbf{k}}^\dagger a_{\mathbf{k}} &=-iV_k^2\left(\frac{1}{\Omega-\omega_k-i0}-\frac{1}{\Omega-\omega_k+i0}\right)b^\dagger b\\ &=2\pi\delta(\Omega-\omega_k)V_\Omega^2 b^\dagger b.\end{aligned} \tag{B14}$$

To obtain the last expression we used Eq. (B13). Replacing Eq. (B14) in Eq. (48) with $O=\int d\mathbf{k}O_{\mathbf{k}}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ we obtain Eq. (50):

$$\frac{d}{dt}(\rho|a_{\mathbf{k}}^\dagger a_{\mathbf{k}})=2\pi V_\Omega^2\delta(\omega_k-\Omega)(\rho|b^\dagger b).$$

Let us deduce the master equation in the thermodynamic limit. In this case, from Eqs. (56) we have the projectors $\mathbb{P}$ and $\mathbb{Q}$, defined by

$$(\rho|\mathbb{P}^\dagger O)\equiv(\mathbb{P}\rho|O)=\rho_1^*O_1+\int d\mathbf{k}\rho_{\mathbf{k}}^*O_{\mathbf{kk}}, \tag{B15}$$

$$\begin{aligned}(\rho|\mathbb{Q}^\dagger O)\equiv(\mathbb{Q}\rho|O)=&\int d\mathbf{k}\int d\mathbf{k}'\rho_{\mathbf{kk}'}^*O_{\mathbf{kk}'}\\ &+\int d\mathbf{k}\rho_{\mathbf{k}1}^*O_{\mathbf{k}1}+\int d\mathbf{k}'\rho_{1\mathbf{k}'}^*O_{1\mathbf{k}'},\end{aligned} \tag{B16}$$

for all $O\in\mathcal{O}_{\rm TD}$, i.e., without the singular part.

Using Eqs. (B16) and (B2)–(B9), we obtain

$$\begin{aligned}\mathbb{L}_1^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}_1^\dagger\mathbb{P}^\dagger b^\dagger b &=\int d\mathbf{k}\ V_k^2\left(\frac{1}{\omega_k-\Omega+i0}+\frac{1}{\Omega-\omega_k+i0}\right)b^\dagger b\\ &-\int d\mathbf{k}\int d\mathbf{p}\ V_kV_p\left(\frac{a_{\mathbf{k}}^\dagger a_{\mathbf{p}}}{\omega_k-\Omega+i0}-\frac{a_{\mathbf{p}}^\dagger a_{\mathbf{k}}}{\Omega-\omega_k+i0}\right).\end{aligned} \tag{B17}$$

From Eqs. (B15) and (B17),

$$\begin{aligned}\left(\rho\middle|\mathbb{P}^\dagger\mathbb{L}_1^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}_1^\dagger\mathbb{P}^\dagger b^\dagger b\right) &=\left(\mathbb{P}\rho\middle|\mathbb{L}_1^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}_1^\dagger\mathbb{P}^\dagger b^\dagger b\right)\\ &=-2\pi i\int d\mathbf{k}\ V_k^2\delta(\omega_k-\Omega)\rho_1^*+2\pi i\\ &\times\int d\mathbf{k}\ V_k^2\delta(\omega_k-\Omega)\rho_{\mathbf{k}}^*.\end{aligned} \tag{B18}$$

For the last equation we use Eq. (B13). Replacing Eq. (B18) in Eq. (48) with $O=b^\dagger b$, we deduce Eq. (57):

$$\frac{d}{dt}\rho_1^*(t)=-8\pi\Omega^2V_\Omega^2\rho_1^*(t)+2\pi V_\Omega^2\int d\mathbf{k}\delta(\omega_k-\Omega)\rho_{\mathbf{k}}^*(t).$$

If we consider $O=\int d\mathbf{k}d\mathbf{k}'O_{\mathbf{kk}'}a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}$ and Eqs. (B7), (B4), (B5), (B8), (B9), and (B16), we obtain

$$\begin{aligned}\mathbb{L}_V^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}_V^\dagger O &=\int d\mathbf{k}\int d\mathbf{k}'O_{\mathbf{kk}'}V_kV_{k'}\\ &\times\left(\frac{1}{\Omega-\omega_{k'}+i0}+\frac{1}{\omega_k-\Omega+i0}\right)b^\dagger b\\ &+\int d\mathbf{k}\int d\mathbf{k}'O_{\mathbf{kk}'}\int d\mathbf{p}\\ &\times\left(\frac{V_pV_ka_{\mathbf{p}}^\dagger a_{\mathbf{k}'}}{\Omega-\omega_{k'}+i0}+\frac{V_{k'}V_pa_{\mathbf{k}}^\dagger a_{\mathbf{p}}}{\omega_k-\Omega+i0}\right).\end{aligned} \tag{B19}$$

From Eqs. (B15) and (B17),

$$\left(\mathbb{L}_V\mathbb{Q}\frac{1}{\mathbb{L}_0-i0}\mathbb{Q}\mathbb{L}_V\mathbb{P}\rho\middle|O\right)$$
$$=\left(\mathbb{P}\rho\middle|\mathbb{L}_V^\dagger\mathbb{Q}^\dagger\frac{1}{\mathbb{L}_0^\dagger+i0}\mathbb{Q}^\dagger\mathbb{L}_V^\dagger O\right)$$
$$=\int\int d\mathbf{k}d\mathbf{k}'V_kV_{k'}O_{\mathbf{k}\mathbf{k}'}$$
$$\times\left\{\left[\frac{1}{\Omega-\omega_k-i0}+\frac{1}{\omega_{k'}-\Omega-i0}\right]\rho_1^*\right.$$
$$\left.+\frac{\rho_{\mathbf{k}'}^*}{\Omega-\omega_{k'}+i0}+\frac{\rho_{\mathbf{k}}^*}{\omega_k-\Omega+i0}\right\}. \tag{B20}$$

Comparing Eq. (B20) with Eq. (B15), we conclude that the singular part

$$\mathbb{P}\left(\mathbb{L}_V\mathbb{Q}\frac{1}{\mathbb{L}_0-i0}\mathbb{Q}\mathbb{L}_V\mathbb{P}\rho\right)$$

is zero, because it is impossible to obtain a factor proportional to $\delta(\mathbf{k}-\mathbf{k}')$ from the expression between curly brackets in Eq. (B20). The master equation gives

$$\left(i\frac{d}{dt}\mathbb{P}\rho\middle|\int d\mathbf{k}d\mathbf{k}'O_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}\right)$$
$$=-i\int d\mathbf{k}\frac{d}{dt}\rho_{\mathbf{k}}^*O_{\mathbf{k}\mathbf{k}}=\left(\mathbb{P}\mathbb{L}_V\mathbb{Q}\frac{1}{i0-\mathbb{L}_0}\mathbb{Q}\mathbb{L}_V\mathbb{P}\rho\middle|O\right)=0$$

and therefore we deduce Eq. (58):

$$\frac{d}{dt}\rho_{\mathbf{k}}^*(t)=0.$$

APPENDIX C: LONG TIME BEHAVIOR OF THE DECAYING PROCESS

In this section we analyze the long time behavior of the decaying term:

$$\langle b^\dagger b\rangle_t=\langle n\rangle_0|a(t)|^2,\quad a(t)=\int\frac{d\mathbf{k}V_k^2e^{i\omega_kt}}{\eta_+(\omega_k)\eta_-(\omega_k)}, \tag{C1}$$

obtained in Eq. (31) for the case of a finite number of excited modes. As in our model $\omega_k=k$, we can use polar coordinates to write

$$a(t)=4\pi\lim_{K\to\infty}\int_0^K e^{ikt}\gamma(k)dk,\quad \gamma(k)\equiv k^2F(k),$$

$$F(k)\equiv\frac{V_k^2}{\eta_+(k)\eta_-(k)}. \tag{C2}$$

Performing three partial integrations in the previous expression, we obtain

$$\int_0^K e^{ikt}\gamma(k)dk=-\frac{i}{t}[e^{ikt}\gamma(k)]_0^K+\frac{1}{t^2}[e^{ikt}\gamma'(k)]_0^K$$
$$+\frac{i}{t^3}[e^{ikt}\gamma''(k)]_0^K-\frac{i}{t^3}\int_0^K e^{ikt}\gamma'''(k)dk, \tag{C3}$$

where

$$\gamma'(k)=2kF(k)+k^2F'(k),$$
$$\gamma''(k)=2F(k)+4kF'(k)+k^2F''(k), \tag{C4}$$
$$\gamma'''(k)=6F'(k)+6kF''(k)+k^2F'''(k).$$

If V_k is a Schwartz function in $\mathbb{R}^+$ (a reasonable choice for the interaction) $F(k)$ and $\gamma(k)$ are also Schwartz functions in $\mathbb{R}^+$ and $\gamma(\infty)=\gamma'(\infty)=\gamma''(\infty)=0$. Therefore, replacing Eq. (C3) in Eq. (C2) we obtain

$$a(t)=\frac{8\pi iF(0)}{t^3}-\frac{i}{t^3}\int_0^\infty e^{ikt}\gamma'''(k)dk.$$

The integral in the last expression vanishes for $t\to\infty$, as a consequence of the Riemann-Lebesgue theorem, and we finally obtain

$$\lim_{t\to\infty}t^3a(t)=8\pi iF(0).$$

As a conclusion, if $V_{k=0}$ does not vanish, $a(t)$ behaves as t^{-3} for $t\to\infty$.

APPENDIX D: ON THE APPLICABILITY OF THE MASTER EQUATION

In Ref. [13], Antoniou and Tasaki gave a perturbative algorithm based on the subdynamics formalism [13,14], from which a generalized spectral decomposition of the Liouville–Von Newmann operator can be obtained. This spectral decomposition is analytic in the interaction parameter. An extension of this formalism to the functional approach was recently used by Id Betan and one of us [23] to discuss the Friedrichs model.

The construction is based on the decomposition of the states through projectors $\mathbb{P}_n^\dagger$ $(n=0,1,2,\ldots)$ onto the degrees of correlation defined by

$$\mathbb{P}_n^\dagger\mathbb{P}_{n'}^\dagger=\delta_{nn'}\mathbb{P}_n^\dagger,\quad \mathbb{L}_0^\dagger\mathbb{P}_n^\dagger=\mathbb{P}_n^\dagger\mathbb{L}_0^\dagger,$$

$$\sum_n\mathbb{P}_n^\dagger=\mathbb{I}^\dagger,\quad \mathbb{P}_0^\dagger(\mathbb{L}_1^\dagger)^n\mathbb{P}_m^\dagger\begin{cases}=0 & \text{if}\quad n<m\\ \neq 0 & \text{if}\quad n=m.\end{cases} \tag{D1}$$

In the last expressions, $\mathbb{P}_0^\dagger$ is the projector onto the invariant part of the states under the action of the free time evolution $(\mathbb{L}_0^\dagger\mathbb{P}_0^\dagger=\mathbb{P}_0^\dagger\mathbb{L}_0^\dagger=0)$, $\mathbb{I}^\dagger$ is the identity operator $(\mathbb{I}^\dagger O=O)$, and $\mathbb{L}_1^\dagger$ is the interaction part of the Liouville–Von Newmann operator $(\mathbb{L}_1^\dagger O\equiv[V,O])$.

Through a nonunitary transformation, the Liouville–Von Newmann operator $\mathbb{L}^\dagger$ is made isospectral to an intermediate operator $\Theta^\dagger$, which is block diagonal in the degrees of correlation

$$\mathbb{L}^\dagger=\Omega^{\dagger-1}\Theta^\dagger\Omega^\dagger,\quad \Theta^\dagger=\sum_n \mathbb{P}_n^\dagger\Theta^\dagger\mathbb{P}_n^\dagger=\sum_n \Theta_n^\dagger,$$

$$\Theta_n^\dagger=\mathbb{P}_n^\dagger\mathbb{L}^\dagger\mathbb{P}_n^\dagger+\mathbb{P}_n^\dagger\mathbb{C}_n^\dagger\mathbb{L}^\dagger\mathbb{P}_n^\dagger,$$

$$\Omega^\dagger=\sum_n (\mathbb{P}_n^\dagger+\mathbb{C}_n^\dagger),\quad \Omega^{\dagger-1}=\sum_n (\mathbb{P}_n^\dagger+\mathbb{D}_n^\dagger)(\mathbb{P}_n^\dagger+\mathbb{C}_n^\dagger\mathbb{D}_n^\dagger)^{-1}. \tag{D2}$$

The operators $\mathbb{C}_n^\dagger$ and $\mathbb{D}_n^\dagger$ (called creation and destruction of correlations) can be obtained by iteration from the equations,

$$\mathbb{C}_n^\dagger\mathbb{P}_m^\dagger=-i\int_0^{\pm\infty}dte^{-i\mathbb{L}_0^\dagger t}(\mathbb{P}_n^\dagger+\mathbb{C}_n^\dagger)\mathbb{L}_1^\dagger(\mathbb{C}_n^\dagger\mathbb{P}_m^\dagger-\mathbb{P}_m^\dagger)e^{i\mathbb{L}_0^\dagger t},$$

$$m\gtrless n,$$

$$\mathbb{P}_m^\dagger\mathbb{D}_n^\dagger=-i\int_0^{\pm\infty}dte^{-i\mathbb{L}_0^\dagger t}(\mathbb{P}_m^\dagger-\mathbb{P}_m^\dagger\mathbb{D}_n^\dagger)\mathbb{L}_1^\dagger(\mathbb{P}_n^\dagger+\mathbb{D}_n^\dagger)e^{i\mathbb{L}_0^\dagger t},$$

$$m\lessgtr n, \tag{D3}$$

starting with $\mathbb{C}_n^{\dagger(0)}=\mathbb{D}_n^{\dagger(0)}=0$. To obtain the previous equations, boundary conditions have been imposed in such a way that the increase of correlations is future oriented.

Once the spectral decomposition of $\Theta^\dagger$ is obtained in the form

$$\Theta^\dagger=\sum_n \Theta_n^\dagger=\sum_n\sum_\alpha z_{n\alpha}|\tilde{u}_{n\alpha})(u_{n\alpha}|,$$

the generalized eigenvectors $|\tilde{f}_{n\alpha})=(\Omega^\dagger)^{-1}|\tilde{u}_{n\alpha})$ and $(f_{n\alpha}|=(u_{n\alpha}|\Omega^\dagger$ of the Liouville–Von Newmann operator can be computed. The time evolution of a state is given by

$$(\rho_t|=\sum_{n\alpha} e^{iz_{n\alpha}t}(\rho_0|\tilde{f}_{n\alpha})(f_{n\alpha}|.$$

As the eigenvectors and eigenvalues are obtained from a perturbative expansion in powers of the interaction parameter, one possible approximated time evolution can be given computing the eigenvectors up to zero order and the eigenvalues up to second order:

$$(\rho_t|\cong\sum_{n\alpha} e^{iz_{n\alpha}^{(2)}t}(\rho_0|\tilde{f}_{n\alpha}^{(0)})(f_{n\alpha}^{(0)}|. \tag{D4}$$

Taking into account that $|\tilde{f}_{n\alpha}^{(0)})=|\tilde{u}_{n\alpha}^{(0)})$ and $(f_{n\alpha}^{(0)}|=(u_{n\alpha}^{(0)}|$, Eq. (D4) gives

$$(\rho_t|\mathbb{P}_0^\dagger\cong\sum_{n\alpha} e^{iz_{n\alpha}^{(2)}t}(\rho_0|\tilde{u}_{n\alpha}^{(0)})(u_{n\alpha}^{(0)}|\mathbb{P}_0^\dagger.$$

As $(u_{n\alpha}^{(0)}|\mathbb{P}_0^\dagger=\delta_{n0}(u_{0\alpha}^{(0)}|$, we obtain

$$(\rho_t|\mathbb{P}_0^\dagger\cong\sum_\alpha e^{iz_{0\alpha}^{(2)}t}(\rho_0|\tilde{u}_{0\alpha}^{(0)})(u_{0\alpha}^{(0)}|=(\rho_0|\exp(i\Theta_0^{\dagger(2)}t).$$

In the last expression, $\Theta_0^{\dagger(2)}$ is the second order approximation for $\Theta_0^\dagger$. It can be computed from Eqs. (D2) and (D3) as

$$\Theta_0^{\dagger(2)}=-\mathbb{P}_0^\dagger\mathbb{L}_1^\dagger\mathbb{Q}_0^\dagger\frac{1}{i0+\mathbb{L}_0^\dagger}\mathbb{Q}_0^\dagger\mathbb{L}_1^\dagger\mathbb{P}_0^\dagger,\quad \mathbb{Q}_0^\dagger\equiv\mathbb{I}^\dagger-\mathbb{P}_0^\dagger.$$

Therefore, if we admit the approximated expression (D4), $(\mathbb{P}_0\rho_t|$ satisfies the master equation

$$-i\frac{d}{dt}(\mathbb{P}_0\rho_t|=(\mathbb{P}_0\rho_t|\Theta_0^{\dagger(2)},$$

which is equivalent to Eq. (42). The master equation is valid when expression (D4) is a good approximation for the time evolution. If this is the case, the interaction parameter λ should be small, but in addition the time cannot be too large, i.e., $\lambda\ll 1$ and $\lambda^3 t\ll 1$, or equivalently

$$\lambda\ll 1,\quad t\lesssim\lambda^{-2}.$$

This result shows why the exact solutions obtained in Sec. IV differ from the solutions of the master equation for $t\to\infty$. Therefore, the approximation of the master equation eliminates the Khalfin effect.

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